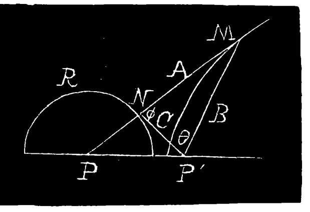
A simple modification leads to a construction for a tangent to a parabola from any external point C. We have only to replace the directing circle by the directrix of the parabola.

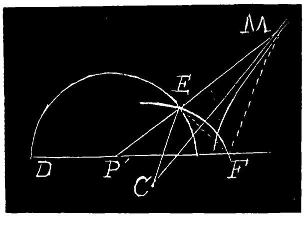
THE HYPERBOLA.

PROPOSITION VI. *If at two fixed points P and P', three lines A, B, and C, be pivoted, A at one point revolving in one direction at any velocity; B and C at the other pivot revolving in an opposite direction, C at such a rate that it constantly intersects A in the circumference of a directing circle described with P as a center, B at such a rate that the angle BC is constantly equal to the angle CA, then the locus of the intersection M of A and B is a hyperbola.*



Let the angle AC be denoted by ϕ and BC by θ. Since $\phi=\theta$, the segment NM=segment $P'M$ in any position. Therefore $PM-P'M=PM-NM=PN=$ constant. Therefore the locus of M is a hyperbola.

PROPOSITION VII. *If a circle with center C be described in the plane of a hyperbola passing through one focus P and intersecting the directing circle at E, and the other focal radius $P'M$ be drawn through this point E to meet the curve at M, the line CM is tangent to the hyperbola.*



Draw MP and EP. The triangle EMP is isosceles and CM is perpendicular to the base PE at its mid-point A. Therefore it passes through the vertex M and is tangent to the hyperbola.

A METHOD FOR CONSTRUCTING AN HYPERBOLA, GIVEN THE ASYMPTOTES AND A FOCUS.

By ARCHIBALD HENDERSON, Ph. D., Associate Professor of Mathematics, University of North Carolina, Chapel Hill, N. C.

Consider any circle, whose center is the point $(0, y_0)$ and whose radius is the distance from this point to the focus $[\sqrt{(a^2+b^2)}, 0]$ of an hyperbola. The equation of this circle is

$$x^2+(y-y_0)^2=y_0^2+a^2+b^2,$$
$$\text{or } x^2+y^2-2y_0y-(a^2+b^2)=0\ldots.(1).$$

Now we may represent any point on an asymptote to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1\ldots.(2)$$

by introducing the parameter t. Thus $(x_1, y_1)\equiv(at_1, bt_1)$ represents any point on the asymptote

$$y-\frac{b}{a}x=0\ldots.(3),$$

and $(x_2, y_2)\equiv(-at_2, bt_2)$ represents any point on the asymptote

$$y+\frac{b}{a}x=0\ldots.(4).$$

If the circle (1) cuts the asymptotes (3) and (4) in the specified points (x_1, y_1), (x_2, y_2), respectively, we have

$$(a^2+b^2)(t_1^2-1)=2by_0t_1\ldots.(5),$$
$$(a^2+b^2)(t_2^2-1)=2by_0t_2\ldots.(6).$$

By division we obtain

$$\frac{t_1^2-1}{t_2^2-1}=\frac{t_1}{t_2},$$

which may be written

$$(t_1-t_2)(t_1t_2+1)=0\ldots.(7).$$

The solution

$$t_1-t_2=0\ldots.(8)$$

shows that, for one position of (x_2, y_2), the line joining (x_1, y_1) and (x_2, y_2) is parallel to the x-axis. Discarding this case, let us consider the solution

$$t_1t_2+1=0\ldots.(9).$$

Since $(x_2, y_2)\equiv\left(\frac{a}{t_1}, \frac{-b}{t_1}\right)$, the equation of the line joining (x_1, y_1) and (x_2, y_2) is

$$\frac{y-bt_1}{\left(\frac{-b}{t_1}-bt_1\right)}=\frac{x-at_1}{\left(\frac{a}{t_1}-at_1\right)},$$

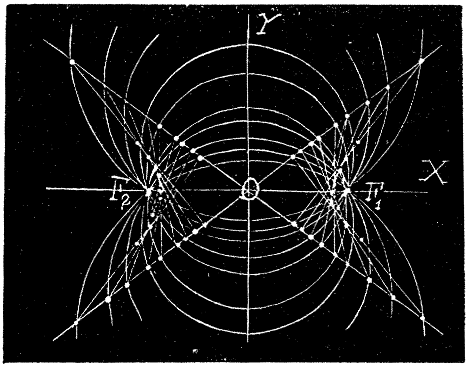
or

$$y=\frac{b}{a}\left(\frac{t_1^2+1}{t_1^2-1}\right)x-\frac{2bt_1}{t_1^2-1}\ldots.(10).$$

But this line touches the hyperbola (2), since

$$\frac{4b^2t_1^2}{(t_1^2-1)^2}\equiv b^2\left(\frac{t_1^2+1}{t_1^2-1}\right)^2-b^2 \ldots\ldots (11).$$

The lines joining the pairs of points (right hand, say) in which a system of co-axial circles, passing through the foci of an hyperbola, cuts the asymptotes, envelope that hyperbola.



Since, moreover, the middle point of the line joining (x_1, y_1), (x_2, y_2) lies on the hyperbola, we have the theorem:

The middle points of the lines joining the pairs of points in which a system of co-axial circles, passing through the foci of an hyperbola, cuts the asymptotes, describe that hyperbola.

These two theorems give two methods for constructing an hyperbola, the one by lines, the other by points, when the asymptotes and a focus are known.* Other constructions might readily have been given, but those given above seem the most instructive.

The University of Chicago, November, 1902.

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* Compare the November number of the MONTHLY for a note by the writer on the converse of this problem.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

163. Proposed by CHRISTIAN HORNUNG, A.M., Professor of Mathematics, Heidelberg University, Tiffin, O.

Three Dutchmen and their wives went to market to buy hogs. The names of the men were Hans, Klaus, and Hendricks, and of the women, Gertrude, Anna, and Katrine; but it was not known which was the wife of each man. They each bought as many hogs as each man or woman paid shillings for each hog, and each man spent three guineas more than his wife. Hendricks bought 23 hogs more than Gertrude, and Klaus bought 11 more than Katrine. What was the name of each man's wife?

Solution by J. SCHEFFER, A. M., Hagerstown, Md., and M. E. GRABER, Heidelberg University, Tiffin, O.

Let x represent the number of one of the women's hogs, and y the number of *her* husband's; then by the conditions of the problem $y^2=x^2+63$. Consequently x^2+63 must be an integer, since $\sqrt{(x^2+63)}$ represents the number of hogs. The equation $y^2-x^2=63$ or $(y+x)(y-x)=63$ admits of three solutions, viz., 63×1, 21×3, and 9×7.